Covariance Control and its Relationship to 17 Other Control Problems

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17 Different Control Design Problem

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

Continuous-Time Case

1. Stabilizing Control
2. Covariance Upper Bound Control
3. Linear Quadratic Regulator
4. *L*∞ Control
5. *H*∞ Control
6. Positive Real Control
7. Robust *H*2 Control
8. Robust *L*∞ Control
9. Robust *H*∞ Control

Discrete-Time Case

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Skelton RE, Iwasaki T, Grigoriadis K (1998): A Unified Algebraic Approach to Control Design. Taylor & Francis, London.

Linear Matrix Equalities and Inequalities

For any given matrices $\boldsymbol{\Gamma}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Theta}$, the LMI :

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

- **Existence Condition**

$$\mathbf{U}_{\boldsymbol{\Gamma}}^{\mathbf{T}}\boldsymbol{\Theta}\boldsymbol{U}_{\boldsymbol{\Gamma}} < \mathbf{0}, \quad \textbf{or} \quad \boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathbf{T}} > \mathbf{0},$$
$$\mathbf{V}_{\Lambda}^{\mathbf{T}}\boldsymbol{\Theta}\boldsymbol{V}_{\boldsymbol{\Lambda}} < \mathbf{0}, \quad \textbf{or} \quad \boldsymbol{\Lambda}^{\mathbf{T}}\boldsymbol{\Lambda} > \mathbf{0}.$$

$\mathbf{U}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B} = \mathbf{0}$, $\mathbf{U}_{\mathbf{B}}^{\mathbf{T}}\mathbf{U}_{\mathbf{B}} > \mathbf{0}$, ($\mathbf{B}\mathbf{V}_{\mathbf{B}} = \mathbf{0}$, $\mathbf{V}_{\mathbf{B}}^{\mathbf{T}}\mathbf{V}_{\mathbf{B}} > \mathbf{0}$) for any matrix $\mathbf{B}$

- **Solutions for G**

If $\mathbf{G}$ exists, then one set of such $\mathbf{G}$ is given by

$$\mathbf{G} = -\rho\boldsymbol{\Gamma}^{\mathbf{T}}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\mathbf{T}}(\boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\mathbf{T}})^{-1}, \boldsymbol{\Phi} = (\rho\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathbf{T}} - \boldsymbol{\Theta})^{-1},$$

where $\rho > 0$ is an arbitrary scalar such that

$$\boldsymbol{\Phi} = (\rho\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathbf{T}} - \boldsymbol{\Theta})^{-1} > \mathbf{0}.$$

Different Interpretations of the Lyapunov Equation

- **Stability Condition**

For any initial condition $\mathbf{x}(0)$ of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\Updownarrow$$

there exists a matrix $\mathbf{Q}$ such that $\mathbf{Q} > \mathbf{0}$ and $\mathbf{Q}\mathbf{A} + \mathbf{A}^{\mathbf{T}}\mathbf{Q} < \mathbf{0}$

- **Controllability Gramian**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \text{ and } \mathbf{y} = \mathbf{C}\mathbf{x}$$

The "Controllability Gramian": $\mathbf{X} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^{\mathbf{T}}e^{\mathbf{A}^{\mathbf{T}}t}dt > \mathbf{0}$ if and only if the pair $(\mathbf{A}, \mathbf{B})$ is controllable

If $\mathbf{X}$ exists, it satisfies $\mathbf{X}\mathbf{A}^{\mathbf{T}} + \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{B}^{\mathbf{T}} = \mathbf{0}$, and $\mathbf{X} > \mathbf{0}$ if and only if $(\mathbf{A}, \mathbf{B})$ is a controllable pair.

Different Interpretations of the Lyapunov Equation

- **Stochastic Interpretations**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w}.$$

$\mathbf{w}$ → zero mean white noise with unit intensity

$$\text{steady-state covariance matrix} \rightarrow \mathbf{X} = \mathbb{E}[\mathbf{x}\mathbf{x}^{\mathbf{T}}]$$

$$\mathbf{X}\mathbf{A}^{\mathbf{T}} + \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{B}^{\mathbf{T}} = \mathbf{0}$$

Steady-state covariance matrix

$$\mathbf{X}'\mathbf{A}^{\mathbf{T}} + \mathbf{A}\mathbf{X}' + \mathbf{B}\mathbf{B}^{\mathbf{T}} < \mathbf{0},\ \mathbf{X} < \mathbf{X}'$$ (Linear Matrix Inequality)

Upper Bound Output Covariance Matrix

$$\mathbf{Y} = \mathbb{E}[\mathbf{y}\mathbf{y}^{\mathbf{T}}] = \mathbf{C}\mathbf{X}'\mathbf{C}^{\mathbf{T}} < \bar{\mathbf{Y}}$$

Different Interpretations of the Lyapunov Equation

- **Deterministic Interpretations**

$$\mathbf{X} = \sum_{i=1}^{n_w} \int_0^\infty \mathbf{x}(i,t)\mathbf{x^T}(i,t)dt, \qquad \mathbf{XA^T} + \mathbf{AX} + \mathbf{BB^T} = \mathbf{0}$$

where $\mathbf{x}(i,t)$ represents the solution of the state equation when only the i^{th} excitation $(\mathbf{w_i} = \delta(t))$ is applied, for the system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bw}$.

$$tr(\mathbf{X}) = \sum_{i=1}^{n_w} \|\mathbf{x}(i,\cdot)\|_{\mathcal{L}_2}^2$$

$$tr(\mathbf{CXC^T}) = \sum_{i=1}^{n_w} \|\mathbf{y}(i,\cdot)\|_{\mathcal{L}_2}^2$$

A time-domain interpretation of the $\mathcal{H}_2$ norm of the transfer matrix $\mathbf{T}(\mathbf{s})$ is also given by $\|\mathbf{T}\|_{\mathcal{H}_2}^{\mathbf{2}} = \sum_{\mathbf{i=1}}^{\mathbf{n_w}} \|\mathbf{y}(\mathbf{i},\cdot)\|_{\mathcal{L}_2}^{\mathbf{2}} = \mathbf{tr}(\mathbf{CXC^T})$.

Control Design Problem

Consider the LTI system

$$\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{p} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{A_p} & \mathbf{D_p} & \mathbf{B_p} \\ \mathbf{C_p} & \mathbf{D_y} & \mathbf{B_y} \\ \mathbf{M_p} & \mathbf{D_z} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x_p} \\ \mathbf{w} \\ \mathbf{u} \end{bmatrix},$$

and a dynamic controller

$$\begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{x}}_\mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{D_c} & \mathbf{C_c} \\ \mathbf{B_c} & \mathbf{A_c} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{x_c} \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{z} \\ \mathbf{x_c} \end{bmatrix},$$

Closed-loop system dynamics form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A_{cl}} & \mathbf{B_{cl}} \\ \mathbf{C_{cl}} & \mathbf{D_{cl}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}.$$

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Stabilizing Control

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

There exists a control of the form $\mathbf{u} = \mathbf{G}\mathbf{x}$ that can stabilize the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\Updownarrow$$

There exists a controller $\mathbf{G}$ such that

$$\mathbf{X} > \mathbf{0}, \quad \left[\ \boldsymbol{\Gamma} \ \middle|\ \boldsymbol{\Lambda}^{\mathbf{T}} \ \middle|\ \boldsymbol{\Theta}\ \right] = \left[\ \mathbf{B} \ \middle|\ \mathbf{X}\mathbf{M}^{\mathbf{T}} \ \middle|\ \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}}\ \right].$$

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Covariance Upper Bound Control

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

Upper bounds on the output covariances $\mathbb{E}[\mathbf{y}\mathbf{y}^{\mathbf{T}}] \leq \bar{\mathbf{Y}}$

$$\Updownarrow$$

There exists a controller $\mathbf{G}$ such that $\mathbf{Y} = \mathbf{C}\mathbf{X}\mathbf{C}^{\mathbf{T}} < \overline{\mathbf{Y}}$

$$\mathbf{X} > \mathbf{0},\ \left[\ \boldsymbol{\Gamma}\ \middle|\ \boldsymbol{\Lambda}^{\mathbf{T}}\ \middle|\ \boldsymbol{\Theta}\ \right] = \left[\begin{array}{c|c|cc} \mathbf{B} & \mathbf{X}\mathbf{M}^{\mathbf{T}} & \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}} & \mathbf{D} \\ \mathbf{0} & \mathbf{E}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} & -\mathbf{I} \end{array}\right]$$

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Linear Quadratic Regulator

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{w} \text{ is the impulsive disturbance } \mathbf{w}(t) = \mathbf{w}_0\delta(t)$$

LQR $\rightarrow \|\mathbf{y}\|_{\mathcal{L}_2} < \gamma$ for any vector $\mathbf{w_0}$ such that $\mathbf{w_0^T}\mathbf{w_0} \leq \mathbf{1}$, and $\mathbf{x_0} = \mathbf{0}$, for given $\gamma > 0$

$$\Updownarrow$$

There exists a matrix $\mathbf{Y} > \mathbf{0}$ and controller $\mathbf{G}$ such that $\|\mathbf{D^T}\mathbf{Y}\mathbf{D}\| < \gamma^2$

$$\left[\ \boldsymbol{\Gamma}\ \middle|\ \boldsymbol{\Lambda}^{\mathbf{T}}\ \middle|\ \boldsymbol{\Theta}\ \right] = \left[\begin{array}{c|c|cc} \mathbf{YB} & \mathbf{M^T} & \mathbf{YA} + \mathbf{A^T}\mathbf{Y} & \mathbf{M^T} \\ \mathbf{H} & \mathbf{0} & \mathbf{M} & -\mathbf{I} \end{array}\right].$$

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$\mathcal{H}_\infty$ Control

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

Let the closed-loop transfer matrix from $\mathbf{w}$ to $\mathbf{y}$ with the controller in be denoted by $\mathbf{T}(\mathbf{s})$:

$$\mathbf{T}(\mathbf{s}) = \mathbf{C_{cl}}(\mathbf{sI} - \mathbf{A_{cl}})^{-\mathbf{1}}\mathbf{B_{cl}} + \mathbf{D_{cl}}.$$

$$\|\mathbf{T}\|_{\mathcal{H}_\infty} = \sup \|\mathbf{T}(j\omega)\| < \gamma \text{ for given } \gamma > 0$$

$$\Updownarrow$$

There exists a controller $\mathbf{G}$ such that $\mathbf{Y} = \mathbf{CXC^T} < \overline{\mathbf{Y}}$

$$\mathbf{X} > \mathbf{0},\ \left[\begin{array}{c|c|c} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^{\mathbf{T}} & \boldsymbol{\Theta} \end{array}\right] = \left[\begin{array}{c|c|ccc} \mathbf{B} & \mathbf{XM^T} & \mathbf{AX} + \mathbf{XA^T} & \mathbf{XC^T} & \mathbf{D} \\ \mathbf{H} & \mathbf{0} & \mathbf{CX} & -\gamma\mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{E^T} & \mathbf{D^T} & \mathbf{F^T} & -\gamma\mathbf{I} \end{array}\right]$$

Skelton RE, Iwasaki T, Grigoriadis K (1998): A Unified Algebraic Approach to Control Design. Taylor & Francis, London.

$\mathcal{L}_\infty$ Control

Characterize the set of all controllers $\mathbf{G}$

$$\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda} + (\boldsymbol{\Gamma}\mathbf{G}\boldsymbol{\Lambda})^{\mathbf{T}} + \boldsymbol{\Theta} < \mathbf{0},$$

$\|\mathbf{y}\|_{\mathcal{L}_\infty} < \gamma$ in the presence of any energy-bounded input $\mathbf{w}(t)$

(i.e., $\int_0^\infty \mathbf{w}^{\mathbf{T}}\mathbf{w}dt \leq 1$)

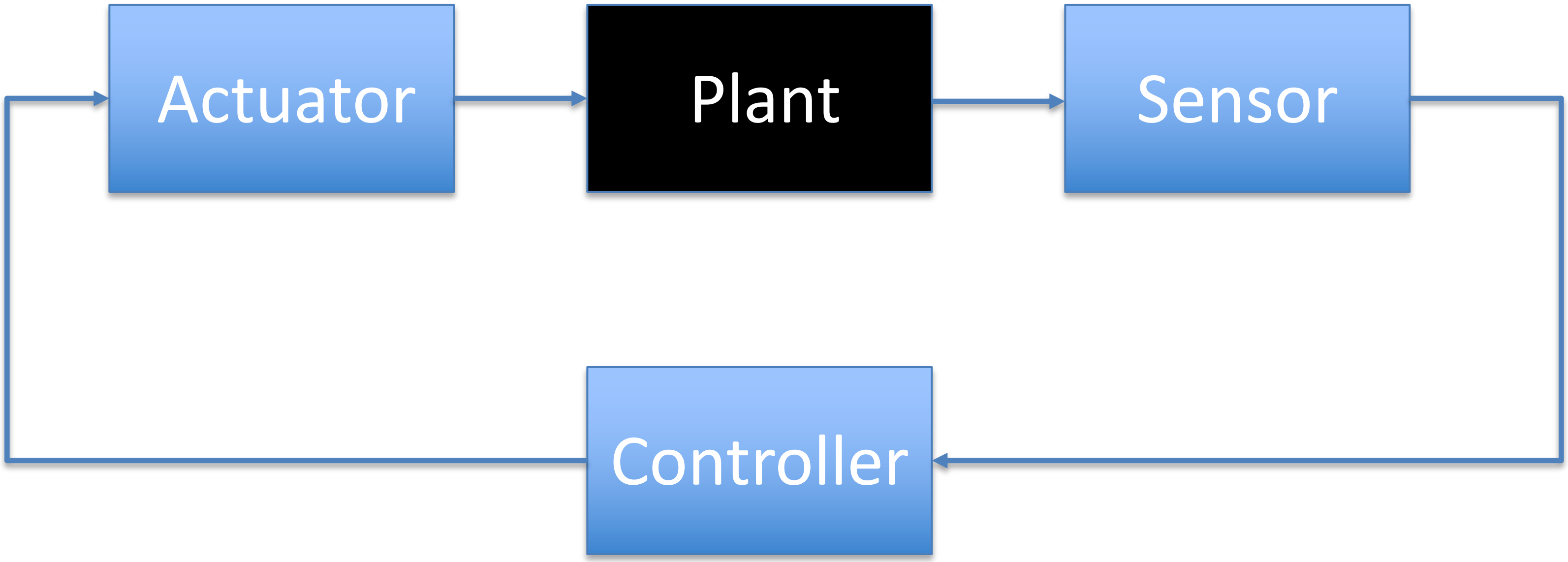
$$\Updownarrow$$

There exists a controller $\mathbf{G}$ such that $\mathbf{C}\mathbf{X}\mathbf{C}^{\mathbf{T}} < \gamma^2\mathbf{I}$

$$\mathbf{X} > \mathbf{0}, \; \left[\begin{array}{c|c|c} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^{\mathbf{T}} & \boldsymbol{\Theta} \end{array}\right] = \left[\begin{array}{c|c|cc} \mathbf{B} & \mathbf{X}\mathbf{M}^{\mathbf{T}} & \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}} & \mathbf{D} \\ \mathbf{0} & \mathbf{E}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} & -\gamma\mathbf{I} \end{array}\right].$$

Skelton RE, Iwasaki T, Grigoriadis K (1998): A Unified Algebraic Approach to Control Design. Taylor & Francis, London.

Integrating Information Architecture and Control



Convex Problem

Faming Li, Mauricio C. de Oliveira, and Robert E. Skelton. "Integrating Information Architecture and Control or Estimation Design". SICE Journal of Control, Measurement, and System Integration, Vol.1(No.2), March 2008.

Motivation

- A true systems design theory would include plant design, appropriate modeling, sensor/actuator selection and control design in a cohesive effort
- A theory such as this may be impossible to develop but there are steps in that direction that are achievable
- Most control problems are defined AFTER sensor and actuator location and precision has been decided
- Defining INFORMATION ARCHITECHTURE (IA) as the selection of instrument type (sensor/actuator), instrument precision (SNR), instrument location, and the control or estimation algorithm, the problem of finding the best IA to meet certain customer requirements can be solved

Problem Statement

- A continuous linear time-invariant system representation:

$$\dot{x} = Ax + Bu + D_p w_p + D_a w_a \quad \text{(Plant)}$$
$$y = Cx \quad \text{(Output)}$$
$$z = Mx + Ew_s \quad \text{(Measurement)}$$

- Noises are modeled as independent zero mean white noises

$$\mathbb{E}_\infty(w_i) = 0, \qquad \mathbb{E}_\infty(w_i w_i^T) = W_i \delta(t-\tau), \qquad i = a, s, p,$$

$$\mathbb{E}_\infty(x) = \lim_{t\to\infty} \mathbb{E}(x)$$

we assume W_p to be known and fixed

- Inverse of noises is defined as precision

Actuator precision

$$\Gamma_a \triangleq W_a^{-1} = \text{diag}(\gamma_a)$$

Sensor precision

$$\Gamma_s \triangleq W_s^{-1} = \text{diag}(\gamma_s)$$

Problem Statement

Total cost for actuators and sensors: $\$ = p_a^T \gamma_a + p_s^T \gamma_s$

p_a - Price for per unit of actuator precision

p_s - Price for per unit of sensor precision

Design a dynamic compensator of the form:

$$\dot{x}_c = A_c x_c + B_c z,$$
$$u = C_c x_c + D_c z,$$

and simultaneously select appropriate actuator and sensor precisions such that the following constraints are satisfied:

$$\$ < \bar{\$}, \quad \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s,$$
$$\mathbb{E}_\infty(uu^T) < \bar{U}, \quad \mathbb{E}_\infty(yy^T) < \bar{Y}$$

for given $\bar{\$}$, $\bar{U}$, $\bar{Y}$, $\bar{\gamma}_a$, $\bar{\gamma}_s$.

Integrating Information Architecture and Control : Final Result

Theorem 1 Let a continuous-time time-invariant linear system be described by the state space equations (1). There exists controller matrices (A_c, B_c, C_c, D_c) such that the cost and performance constraints (3) are satisfied if and only if there exist symmetric matrices X, Y, vectors γ_a, γ_s, and matrices L, F, and Q such that

$$p_a^T\gamma_a + p_s^T\gamma_s < \bar{\$}, \qquad \gamma_a < \bar{\gamma}_a, \qquad \gamma_s < \bar{\gamma}_s, \tag{4}$$

$$\begin{bmatrix} \bar{Y} & CX & C \\ (\cdot)^T & X & I \\ (\cdot)^T & (\cdot)^T & Y \end{bmatrix} > 0, \qquad \begin{bmatrix} \bar{U} & L & 0 \\ (\cdot)^T & X & I \\ (\cdot)^T & (\cdot)^T & Y \end{bmatrix} > 0, \tag{5}$$

$$\begin{bmatrix} \mathrm{Sym}(\Phi_{11}) & \Phi_{12} \\ (\cdot)^T & \Phi_{22} \end{bmatrix} < 0, \tag{6}$$

where

$$\Phi_{11} := \begin{bmatrix} AX + BL & A \\ Q & YA + FM \end{bmatrix}, \quad \Phi_{22} := \begin{bmatrix} -W_p^{-1} & 0 & 0 \\ 0 & -\Gamma_a & 0 \\ 0 & 0 & -\Gamma_s \end{bmatrix}.$$

$$\Phi_{21} := \begin{bmatrix} D_p & D_a & 0 \\ YD_p & YD_a & FE \end{bmatrix},$$

Integrating Information Architecture and Control : Final Result

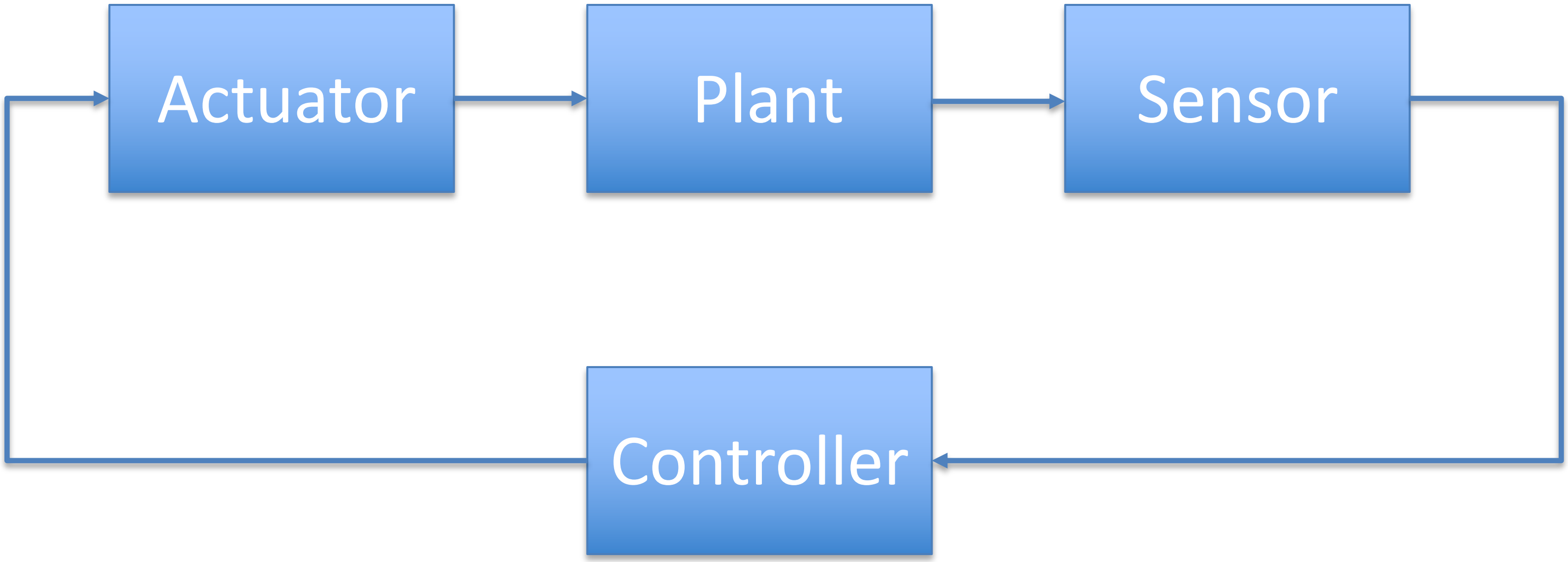
Which produces the control given in the paper:

If the above LMI has a feasible solution then suitable controller matrices can be computed from the solution as

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB \\ 0 & I \end{bmatrix} \times \begin{bmatrix} Q - YAX & F \\ L & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -MXU^{-1} & I \end{bmatrix}, \quad (7)$$

where V and U are nonsingular square matrices satisfying $YX + VU = I$.

Integrated Plant, Sensor/Actuator and Control Design



Non - Convex Problem

New Contribution - Jointly optimize controller, Sensor/Actuator Design and Plant Parameters in an LMI framework to meet some performance criteria.

Integrated Plant, Sensor/Actuator and Control Design

- **Existing Theory**
  - Integrated Structure and control design – **ISCD Paper***
    - Fix structure parameter then design controller
    - Fix Controller and then redesign structure
  - Optimize sensor/actuator precision jointly with control design – **IA Paper****

- **New Contribution**
  - Controller does not need to be fixed in the structure redesign step
  - LMI framework
  - Ability to optimize the mass matrix
  - Jointly optimize controller, Sensor/Actuator Design and Plant Parameters in an LMI framework to meet some performance criteria.

*K. M. Grigoriadis and R. E. Skelton. Integrated structural and control design for vector second-order systems via LMIs. In Proceedings of the 1998 American Control Conference. ACC (IEEE Cat. No.98CH36207), volume 3, pages 1625–1629 vol.3, June 1998.
**Faming Li, Mauricio C. de Oliveira, and Robert E. Skelton. "Integrating Information Architecture and Control or Estimation Design". SICE Journal of Control, Measurement, and System Integration, Vol.1(No.2), March 2008.

Given Data

- A continuous linear time-invariant system in *descriptor* state-space representation:

$$E(\alpha)\dot{x} = A(\alpha)x + Bu + D_p(\alpha)w_p + D_a(\alpha)w_a, \quad \text{(Plant)}$$

$$y = C_y(\alpha)x, \quad \text{(Output)}$$

$$z = C_z x + D_s w_s, \quad \text{(Measurement)}$$

- All these matrices are affine in parameters α $\quad A(\alpha) = A_0 + \sum_i \alpha_i A_i, \quad E(\alpha) = E_0 + \sum_i \alpha_i E_i$

- Noises are modeled as independent zero mean white noises

$$\mathbb{E}_\infty(w_i) = 0, \qquad \mathbb{E}_\infty(w_i w_i^T) = W_i \delta(t - \tau), \qquad i = a, s, p,$$

$$\mathbb{E}_\infty(x) = \lim_{t\to\infty} \mathbb{E}(x)$$ we assume W_p to be known and fixed

Goyal R, Skelton RE,(2019) Joint Optimization of Plant, Controller, and Sensor/Actuator Design. In: Proceedings of the 2019 American control conference, Philadelphia.

Given Data

- Actuator and sensor precisions are defined to be inversely proportional to the respective noise intensities

$$\Gamma_a \triangleq W_a^{-1} = \text{diag}(\gamma_a) \qquad \Gamma_s \triangleq W_s^{-1} = \text{diag}(\gamma_s)$$

- Assuming p_a, p_s and p_α are vectors containing the price per unit of actuator precision, sensor precision and structure parameter

- The total design price : $\$ = p_a^T \gamma_a + p_s^T \gamma_s + p_\alpha^T \alpha,$

Information Architecture System Design

Problem Statement

Design a dynamic compensator of the form:

$$\dot{x}_c = A_c x_c + B_c z,$$
$$u = C_c x_c + D_c z,$$

and simultaneously select the plant parameter values, appropriate actuator and sensor precisions such that the following constraints are satisfied:

$$\$ < \bar{\$}, \quad \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s, \quad \bar{\alpha}_L < \alpha < \bar{\alpha}_U$$
$$\mathbb{E}_\infty(uu^T) < \bar{U}, \quad \mathbb{E}_\infty(yy^T) < \bar{Y}$$

for given $\bar{\$}$, $\bar{U}$, $\bar{Y}$, $\bar{\gamma}_a$, $\bar{\gamma}_s$, $\bar{\alpha}_L$, and $\bar{\alpha}_U$.

Closed Loop System

- Define the closed-loop state and noise vectors as

$$\tilde{x} \triangleq \begin{pmatrix} x \\ x_c \end{pmatrix}, \qquad w \triangleq \begin{pmatrix} w_p \\ w_a \\ w_s \end{pmatrix}.$$

- The closed-loop system is given by

$$E_{cl}\dot{\tilde{x}} = A_{cl}\tilde{x} + B_{cl}w$$
$$y = C_{cl}\tilde{x}$$
$$u = M_{cl}\tilde{x} + F_{cl}w$$

- All the matrices can be expanded as

$$W = \begin{bmatrix} W_p & 0 & 0 \\ 0 & W_a & 0 \\ 0 & 0 & W_s \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A(\alpha) & BC_c \\ B_cC_z & A_c \end{bmatrix},$$
$$E_{cl} = \begin{bmatrix} E(\alpha) & 0 \\ 0 & I_n \end{bmatrix}, B_{cl} = \begin{bmatrix} D_p(\alpha) & D_a(\alpha) & 0 \\ 0 & 0 & B_cD_s \end{bmatrix},$$
$$C_{cl} = \begin{bmatrix} C_y(\alpha) & 0 \end{bmatrix}, \quad M_{cl} = \begin{bmatrix} 0 & C_c \end{bmatrix},$$

Control Design Problem

- Defining $\bar{A}_{cl} = E_{cl}^{-1} A_{cl}$ and $\bar{B}_{cl} = E_{cl}^{-1} B_{cl}$

$$\dot{\bar{x}} = \bar{A}_{cl} x + \bar{B}_{cl} w.$$

- The above closed loop system is stable if and only if there exists a $X > 0$ such that:

$$\bar{A}_{cl} X + X \bar{A}_{cl}^T + \bar{B}_{cl} W \bar{B}_{cl}^T < 0.$$

- Applying Schur's Compliment and defining

$$\delta \triangleq (A_c, B_c, C_c, \gamma_a, \gamma_s, \alpha, Q), \; Q \triangleq X^{-1}$$

$$\mathbb{F}(\delta) \triangleq \begin{bmatrix} -(A_{cl} - E_{cl}) X (A_{cl} - E_{cl})^T & B_{cl} & A_{cl} & E_{cl} \\ B_{cl}^T & -W^{-1} & 0 & 0 \\ A_{cl}^T & 0 & -Q & 0 \\ E_{cl}^T & 0 & 0 & -Q \end{bmatrix} < 0,$$

Not an LMI

(Non-Convex Constraints)

Information Architecture System Design

Existence Condition

Existence Theorem:-

If there exists a matrix X that satisfies all these equations, then the design specifications can be met, and the closed loop system will be stable

$$p_a^T \gamma_a + p_s^T \gamma_s + p_\alpha^T \alpha < \bar{\$}, \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s, \quad \bar{\alpha}_L < \alpha < \bar{\alpha}_U$$

$$\begin{bmatrix} \bar{U} & M_{cl} \\ {M_{cl}}^T & Q \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{Y} & C_{cl} \\ {C_{cl}}^T & Q \end{bmatrix} > 0 \quad Q \triangleq X^{-1}$$

$$\mathbb{F}(\delta) \triangleq \begin{bmatrix} -(A_{cl} - E_{cl})X(A_{cl} - E_{cl})^T & B_{cl} & A_{cl} & E_{cl} \\ {B_{cl}}^T & -W^{-1} & 0 & 0 \\ {A_{cl}}^T & 0 & -Q & 0 \\ {E_{cl}}^T & 0 & 0 & -Q \end{bmatrix} < 0,$$ Not an LMI

$$\Updownarrow$$

$$\$ < \bar{\$}, \quad \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s, \quad \bar{\alpha}_L < \alpha < \bar{\alpha}_U \quad \mathbb{E}_\infty(uu^T) < \bar{U}, \quad \mathbb{E}_\infty(yy^T) < \bar{Y}$$

Design Specifications

Convexifying Algorithm Lemma

A stationary point of the non-convex optimization problem

$$\bar{\delta} = \arg\min_{\delta \in \Omega} f(\delta), \quad \Omega = \{\delta \in \phi | \mathbb{F}(\delta) < 0\} \tag{1}$$

can be obtained by iterating over a sequence of convex subproblems given by

$$\bar{\delta}_{k+1} = \arg\min_{\delta \in \Omega_k} f(\delta), \Omega_k = \{\delta \in \phi | \mathbb{F}(\delta) + \mathbb{G}(\delta, \delta_k) < 0\}. \tag{2}$$

where the potential function $\mathbb{G} \geq 0$ and $\mathbb{G}(\delta, \eta) = 0$ if and only if $\delta = \eta$.

Juan F. Camino, M. C. de Oliveira, and R. E. Skelton, ‘‘Convexifying’’ Linear Matrix Inequality Methods for Integrating Structure and Control Design, J. Struct. Eng., 2003, 129(7): 978-988

Convexifying the Problem

- To use the previous Lemma, let us define the matrix *G* as:

$$G(\delta) \triangleq (A_{cl} - E_{cl})X$$

- Also define the convexifying potential function as:

$$\mathbb{G}(\delta,\eta) \triangleq \begin{bmatrix} (A_{cl} - E_{cl} - G(\eta)Q)X(A_{cl} - E_{cl} - G(\eta)Q)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \geq 0$$

The potential function $\mathbb{G} \geq 0$ and $\mathbb{G}(\delta,\eta) = 0$ if and only if $\delta = \eta$.

- The matrix function $\mathbb{F}(\delta) + \mathbb{G}(\delta,\eta)$:

$$\begin{bmatrix} -(A_{cl} - E_{cl})G^T - G(A_{cl} - E_{cl})^T + GQG^T & B_{cl} & A_{cl} & E_{cl} \\ {B_{cl}}^T & -W^{-1} & 0 & 0 \\ {A_{cl}}^T & 0 & -Q & 0 \\ {E_{cl}}^T & 0 & 0 & -Q \end{bmatrix} < 0$$

LMI

(Convex Constraints)

Control Design Problem

If there exists a matrix Q such that the iteration on the following LMIs converges, then all the design objectives can be met, and the closed loop system will be stable

$$p_a^T \gamma_a + p_s^T \gamma_s + p_\alpha^T \alpha < \bar{\$}, \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s, \quad \bar{\alpha}_L < \alpha < \bar{\alpha}_U$$

$$\begin{bmatrix} \bar{U} & M_{cl} \\ {M_{cl}}^T & Q \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{Y} & C_{cl} \\ {C_{cl}}^T & Q \end{bmatrix} > 0$$

$$\begin{bmatrix} -(A_{cl} - E_{cl})G^T - G(A_{cl} - E_{cl})^T + GQG^T & B_{cl} & A_{cl} & E_{cl} \\ {B_{cl}}^T & -W^{-1} & 0 & 0 \\ {A_{cl}}^T & 0 & -Q & 0 \\ {E_{cl}}^T & 0 & 0 & -Q \end{bmatrix} < 0$$

$$\$ < \bar{\$}, \quad \gamma_a < \bar{\gamma}_a, \quad \gamma_s < \bar{\gamma}_s, \quad \bar{\alpha}_L < \alpha < \bar{\alpha}_U \quad \mathbb{E}_\infty(uu^T) < \bar{U}, \quad \mathbb{E}_\infty(yy^T) < \bar{Y}$$

Design Specifications

Goyal R, Skelton RE,(2019) Joint Optimization of Plant, Controller, and Sensor/Actuator Design. In: Proceedings of the 2019 American control conference, Philadelphia.

Optimization Versions of the Design Problem

Let four parameters out of the set $(\bar{\alpha}_L, \bar{\alpha}_U, \bar{\$}, \bar{U}, \bar{Y})$ be hard constraints and let the fifth parameter, denoted $\bar{z}$.

Extrema-Finding Algorithm

- Set fixed nominal values for $\bar{z}_0$ and α_0. Compute controller matrices $A_{c,0}$, $B_{c,0}$, $C_{c,0}$, precision vectors $\gamma_{a,0}$, $\gamma_{s,0}$ and inverse covariance matrix Q_0 according to "IA Paper"
- **Repeat:** Set $G_k \leftarrow (A_{cl}(\alpha_k) - E_{cl}(\alpha_k))Q_k^{-1}$
  -For fixed $G = G_k$, find the extrema of $\bar{z}$ for the given feasible LMIs
- **Until:** $\|\bar{z}_k - \bar{z}_{k-1}\| < \epsilon$

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Thank You!

https://bobskelton.github.io/index.html

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