Covariance Steering as a Tool for Planning and Control in the Presence of Uncertainty

Panagiotis Tsiotras

School of Aerospace Engineering
Institute for Robotics and Intelligent Machines
Georgia Institute of Technology

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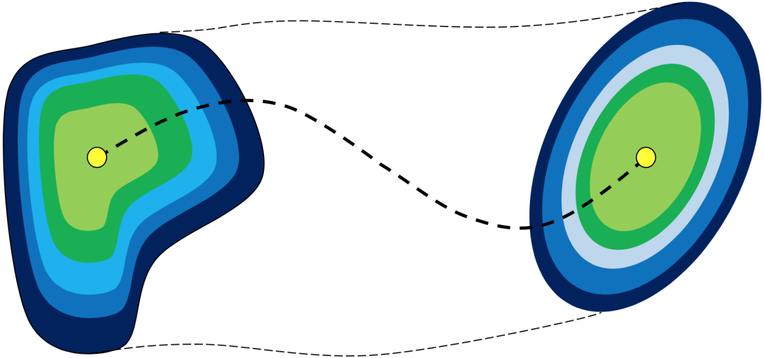
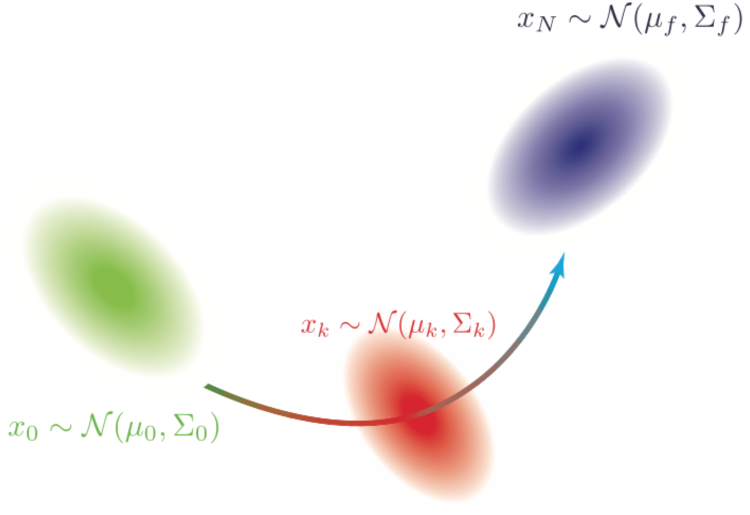
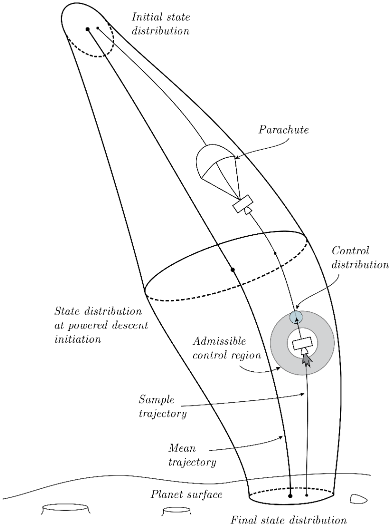
Moving Densities

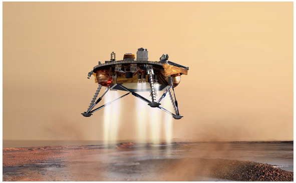


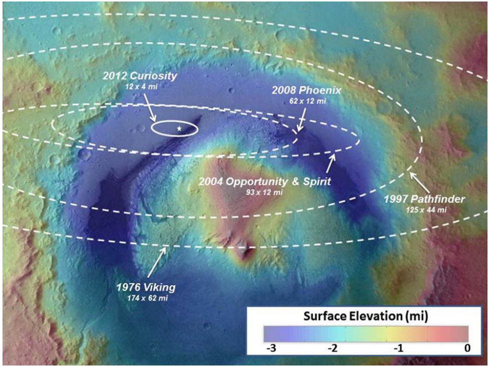
Gaussian Case: Steering the Covariance



Example: Powered Descend Guidance







Problem Formulation

- Given the discrete-time stochastic linear system

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k$$

- Initial and final states to be distributed according to

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \qquad x_N \sim \mathcal{N}(\mu_f, \Sigma_f)$$

  with $\mu_0, \Sigma_0, \mu_f, \Sigma_f$ given.

- Minimize the cost function

$$J(u_0, \ldots, u_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} u_k^\top u_k\right] \rightarrow \min$$

The system state at step $k+1$ is given by

$$x_{k+1} = \mathcal{A}_k x_0 + \mathcal{B}_k U_k + \mathcal{D}_k W_k.$$

where

$$U_k = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \end{bmatrix}, \qquad W_k = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_k \end{bmatrix}$$

and where

$$\mathcal{A}_k \triangleq A_{k,0}, \quad \mathcal{B}_k \triangleq \mathcal{B}_{k,0}, \quad \mathcal{D}_k \triangleq \mathcal{D}_{k,0}$$

$$\mathcal{B}_{k_1,k_0} \triangleq \begin{bmatrix} B_{k_1,k_0} & B_{k_1,k_0+1} & \cdots & B_{k_1,k_1} \end{bmatrix},$$
$$\mathcal{D}_{k_1,k_0} \triangleq \begin{bmatrix} D_{k_1,k_0} & D_{k_1,k_0+1} & \cdots & D_{k_1,k_1} \end{bmatrix},$$

$$A_{k_1,k_0} = A_{k_1} A_{k_1-1} \cdots A_{k_0}, \qquad B_{k_1,k_0} = A_{k_1,k_0+1} B_{k_0}, \qquad D_{k_1,k_0} = A_{k_1,k_0+1} D_{k_0}$$

Let

$$\mathcal{A} = \mathcal{A}_{N-1}, \qquad \mathcal{B} = \mathcal{B}_{N-1}, \qquad \mathcal{D} = \mathcal{D}_{N-1}$$

and

$$U = U_{N-1} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \qquad W = W_{N-1} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}$$

then

$$x_N = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W$$

The mean of the state $\mu_k = \mathbb{E}\left[x_k\right]$ obeys the expression

$$\mu_{k+1} = \mathcal{A}_k \mu_0 + \mathcal{B}_k \overline{U}_k$$

where $\overline{U}_k = \mathbb{E}\left[U_k\right]$. Let

$$\widetilde{U}_k \triangleq U_k - \overline{U}_k, \qquad \widetilde{x}_k \triangleq x_k - \mu_k,$$

It follows that

$$\widetilde{x}_{k+1} = \mathcal{A}_k \widetilde{x}_0 + \mathcal{B}_k \widetilde{U}_k + \mathcal{D}_k W_k.$$

$$J(U) = \mathbb{E}\left[U^\top U\right] = \underbrace{\overline{U}^\top \overline{U}}_{J_\mu} \; + \; \underbrace{\mathtt{tr}\left[\mathbb{E}\left[\widetilde{U}\widetilde{U}^\top\right]\right]}_{J_\Sigma}.$$

Steering the Mean

Main Result

The optimal control $\overline{U}^{\star}$ that minimizes the cost

$$J_\mu = \overline{U}^{\top}\overline{U} = \sum_{k=0}^{N-1} \mathbb{E}\left[u_k\right]^{\top} \mathbb{E}\left[u_k\right]$$

subject to the constraint

$$\mathcal{A}\mu_0 + \mathcal{B}\overline{U} = \mu_f$$

is given by

$$\overline{U}^{\star} = \mathcal{B}^{\top}(\mathcal{B}\mathcal{B}^{\top})^{-1}(\mu_f - \mathcal{A}\mu_0)$$

Diffusionless Case ($D_k = 0$)

Theorem (Goldshtein and Tsiotras, 2017)

Let

$$V_0 S_0 V_0^\top = \Sigma_0, \quad V_F S_F V_F^\top = \Sigma_f,$$

and

$$U_\Omega S_\Omega V_\Omega^\top \triangleq S_F{}^{\frac{1}{2}} V_F^\top (\mathcal{B}\mathcal{B}^\top)^{-1} \mathcal{A} V_0 S_0{}^{\frac{1}{2}}.$$

Then the optimal control gain L that minimizes J_Σ subject to a constraint $\Sigma_N = \Sigma_f$, is given by

$$L^\star = \mathcal{B}^\top (\mathcal{B}\mathcal{B}^\top)^{-1} (V_F S_F{}^{\frac{1}{2}} U_\Omega V_\Omega^\top S_0{}^{-\frac{1}{2}} V_0^\top - \mathcal{A})$$

Control is of the form

$$\widetilde{U} = L \widetilde{x}_0,$$

General Case ($D_k \neq 0$)

Key Observation

The system

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k$$

at time step N can be viewed as a sum of N uncorrelated

$$\mathbb{E}\left[x_k^{(i)} x_m^{(j)\top}\right] = 0, \qquad k, m, i, j \in \{0, \ldots, N\}, \quad i \neq j,$$

diffusion-less sub-systems

$$x_N = \sum_{i=0}^{N-1} x_N^{(i)} + D w_{N-1},$$

$$x_{k+1}^{(i)} = A_k x_k^{(i)} + B_k u_k^{(i)}, \quad x_i^{(i)} = \begin{cases} x_0, & \text{for } i = 0, \\ D_{i-1} w_{i-1}, & \text{otherwise.} \end{cases}$$

Optimal Controller

$$U_{i,N-1}^{(i)} = \begin{cases} L^{(i)} x_i^{(i)}, & i = 1, \ldots, N-1, \\ L^{(0)} x_0 + \mathbb{E}\left[U\right], & i = 0. \end{cases}$$

where

$$U_{k_1,k_2}^{(i)} \triangleq \begin{bmatrix} u_{k_1}^{(i)} \\ u_{k_1+1}^{(i)} \\ \vdots \\ u_{k_2}^{(i)} \end{bmatrix}, \quad 0 \leq k_1 \leq k_2 \leq N-1.$$

Optimal Controller

Assume $\Sigma_0 \succeq 0$ and $\Sigma_f \succeq 0$, let $y_0 = x_0 - \mu_0$, and define

$$y_k = D_{k-1}w_{k-1} = x_k - (A_{k-1}x_{k-1} + B_{k-1}u_{k-1})$$

Let $\Phi_k = (\mathsf{I} + \mathcal{B}_{N,k}\mathcal{B}_{N,k}^\top \Lambda)^{-1} A_{N,k}$, with $\Lambda = \Lambda^\top$ be the solution of the matrix equation

$$\sum_{k=1}^{N-1} \Phi_k D_{k-1} D_{k-1}^\top \Phi_k^\top + \Phi_0 \Sigma_0 \Phi_0^\top = \Sigma_f - D_{N-1} D_{N-1}^\top \succeq 0$$

The optimal linear control law is given by

$$u_k^\star = B_{N,k}^\top (\mathcal{B}\mathcal{B}^\top)^{-1}(\mu_f - \mathcal{A}\mu_0) + \sum_{i=0}^{k} L_k^{(i)} y_i$$

where, $L_k^{(i)} = -B_{N,k}^\top \Lambda \Phi_i$.

Relation with LQG

Theorem (Goldshtein and Tsiotras, 2017; Chen et al, 2016)

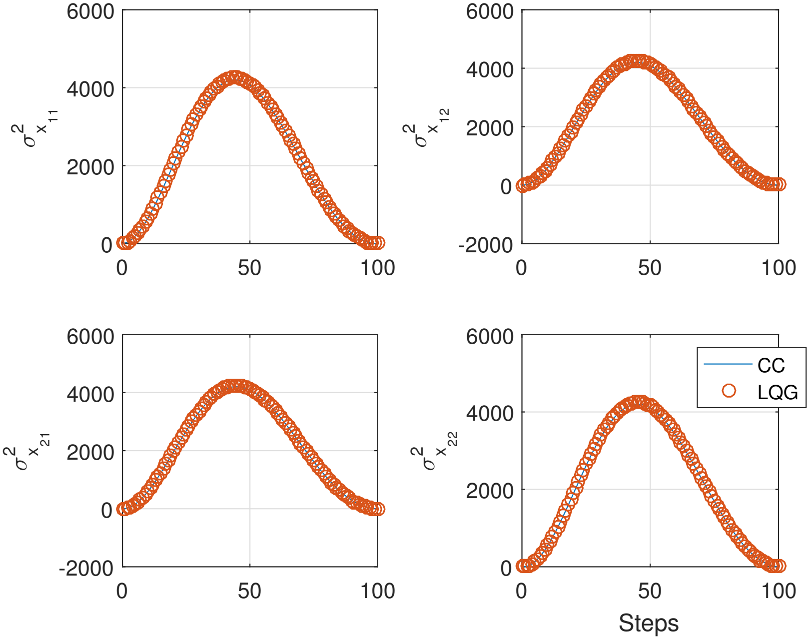
Let initial and final state covariance matrices Σ_0 and Σ_f and symmetric matrix Q_f. Assume that the LQG controller that minimizes the cost function

$$J(u_0, \ldots, u_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} u_k^\top u_k + x_N^\top Q_f x_N\right],$$

results in the final state covariance being equal to Σ_f. Then, this controller coincides is the same as the covariance steering controller with boundary constraints

$$x_0 \sim \mathcal{N}(0, \Sigma_0), \qquad x_N \sim \mathcal{N}(0, \Sigma_f),$$

with $\Lambda = Q_f$.


6000
4000
2000
0
0
50
100
$\sigma^2_{x_{11}}$
$\sigma^2_{x_{12}}$
-2000
$\sigma^2_{x_{21}}$
$\sigma^2_{x_{22}}$
CC
LQG
Steps

General Cost

Consider discrete-time stochastic linear system

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k$$

- We wish the initial and final states to be distributed according to

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \qquad x_N \sim \mathcal{N}(\mu_N, \Sigma_N)$$

where $\mu_0, \Sigma_0, \mu_N, \Sigma_N$ given, while minimizing the cost function

$$J(x,u) = \mathbb{E}\left[\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k\right]$$

where $Q_k \succeq 0$ and $R_k \succ 0$ for all $k = 0, 1, \ldots, N-1$.

- Assume that $\Sigma_0 \succeq 0$ and $\Sigma_N \succ 0$,

- Introduce augmented state $X = [x_0^\top, x_1^\top, \dots, x_N^\top]^\top$ to write

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W$$

cost

$$J(X, U) = \mathbb{E}\left[X^\top \bar{Q} X + U^\top \bar{R} U\right]$$

boundary conditions

$$\mu_0 = E_0 \mathbb{E}[X], \qquad \Sigma_0 = E_0 \left(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\right) E_0^\top$$

$$\mu_N = E_N \mathbb{E}[X], \qquad \Sigma_N = E_N \left(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\right) E_N^\top$$

where E_k is a matrix such that

$$x_k = E_k X, \qquad k = 0, 1, \dots, N$$

Note: $$\mathcal{A} \leftarrow E_N \mathcal{A}, \qquad \mathcal{B} \leftarrow E_N \mathcal{B}, \qquad \mathcal{D} \leftarrow E_N \mathcal{D}$$

- Let the control sequence

$$u_k = v_k + K_k y_k$$

where y_k is given by

$$y_{k+1} = A_k y_k + D_k w_k$$
$$y_0 = x_0 - \mu_0$$

and let the control law

$$U = V + KY$$

Theorem (Okamoto & PT, 2018)

The cost function takes the form

$$J(V, K) = \mathtt{tr}\left(((I + \mathcal{B}K)^\top \bar{Q}(I + \mathcal{B}K) + K^\top \bar{R}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)\right)$$
$$+ (\mathcal{A}\mu_0 + \mathcal{B}V)^\top \bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}V) + V^\top \bar{R}V$$

In addition, the terminal state constraints can be written as

$$\mu_N = E_N\left(\mathcal{A}\mu_0 + \mathcal{B}V\right),$$
$$\Sigma_N = E_N(I + \mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)(I + \mathcal{B}K)^\top E_N^\top$$

Note that V steers the mean and K steers the covariance, respectively.
Letting

$$\Sigma_N \succeq E_N(I + \mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)(I + \mathcal{B}K)^\top E_N^\top$$

yields a **convex problem**.

- Can handle convex chance constraints of the form

$$\texttt{Pr}(x_k \notin \chi) \leq P_{\text{fail}}, \qquad k = 0, \ldots, N-1$$

where

$$\chi = \bigcap_{j=1}^{M} \{x : \alpha_j^\top x \leq \beta_j\}$$

using the standard trick

$$\texttt{Pr}(\alpha_j^\top x \leq \beta_j) = \Phi\left(\frac{\beta_j - a_j^\top x}{\sqrt{\alpha_j^\top \Sigma_x \alpha_j}}\right) \geq 1 - p_j, \qquad \sum_{j=1}^{M} p_j \leq P_{\text{fail}}$$

or

$$\alpha_j^\top x - \beta_j + \sqrt{\alpha_j^\top \Sigma_x \alpha_j}\, \Phi^{-1}(1 - p_j) \leq 0$$

where Φ is the cumulative distribution function of the standard normal distribution.

- Assuming

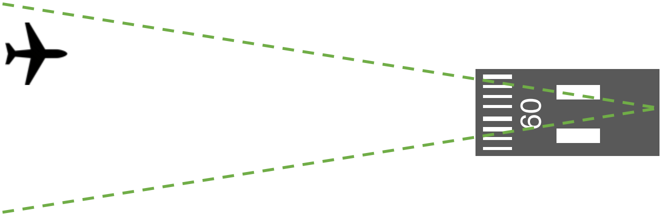
$$\mathtt{Pr}(\alpha_j^\top X > \beta_j) \leq p_{j,\mathrm{fail}} \qquad \sum_{j=1}^{M} p_{j,\mathrm{fail}} \leq P_{\mathrm{fail}}$$

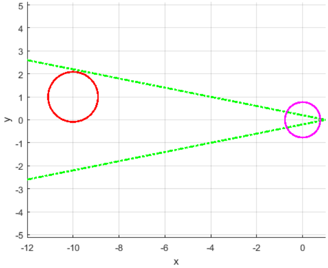
the chance constraint can be formulated as

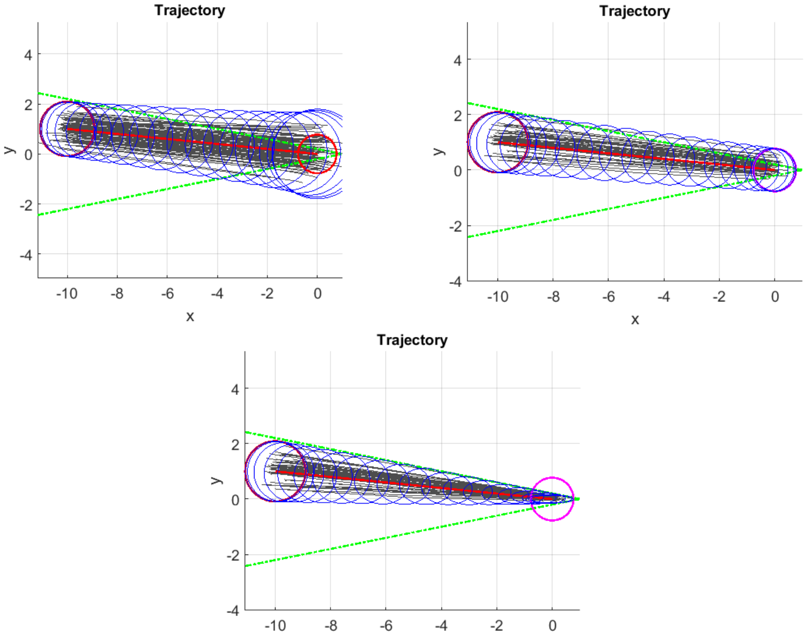
$$\alpha_j^\top (\mathcal{A}\mu_0 + \mathcal{B}V) + \|(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)^{1/2}(I + \mathcal{B}K)^\top \alpha_j\| \Phi^{-1}(1 - p_{j,\mathrm{fail}}) - \beta_j \leq 0$$

Second order cone (convex) constraint in K and V.

Example







Trajectory
y
x
4
2
0
-2
-4
-10
-8
-6
-4
-2
0
Trajectory
y
x
Trajectory
y

Non-Convex Constraints

- For non-convex polytopic constraints, write

$$\chi = \bigcup_{r=0}^{N_R-1} \underbrace{\bigcap_{q=0}^{M_r-1} \{x : \alpha_{r,q}^\top x \leq \beta_{r,q}\}}_{\mathcal{R}_r}$$

  and enforce $\mathtt{Pr}\left(x_k \notin \mathcal{R}_r\right) < \epsilon$ and $\mathtt{Pr}\left(x_{k+1} \notin \mathcal{R}_r\right) < \epsilon$

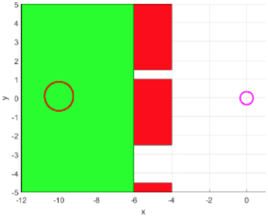
Lemma

Given $\mathcal{R}_r$, the condition

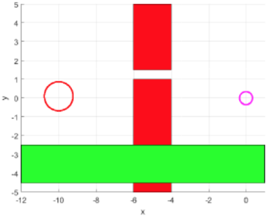
$$\mathtt{Pr}\left(x_k \notin \mathcal{R}_r\right) < \epsilon \quad \text{and} \quad \mathtt{Pr}\left(x_{k+1} \notin \mathcal{R}_r\right) < \epsilon,$$

is a second-order cone constraint in V and K.

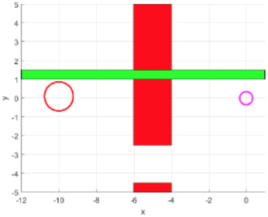
Example



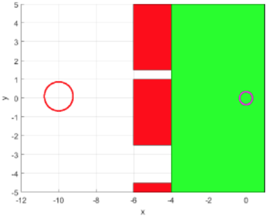
(a) Region 1



(b) Region 2



(c) Region 3

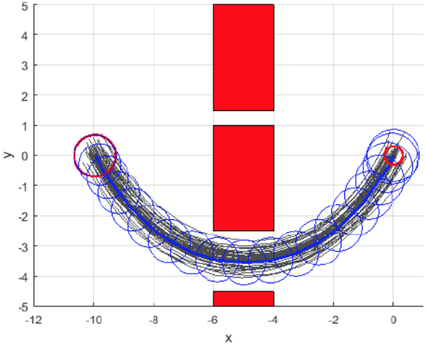


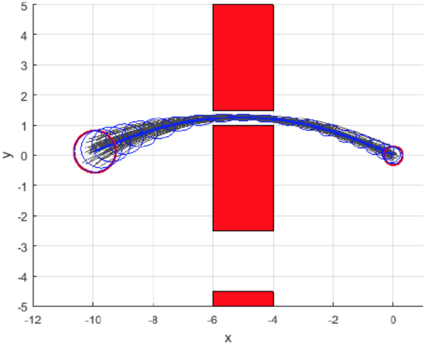
(d) Region 4

N = 20

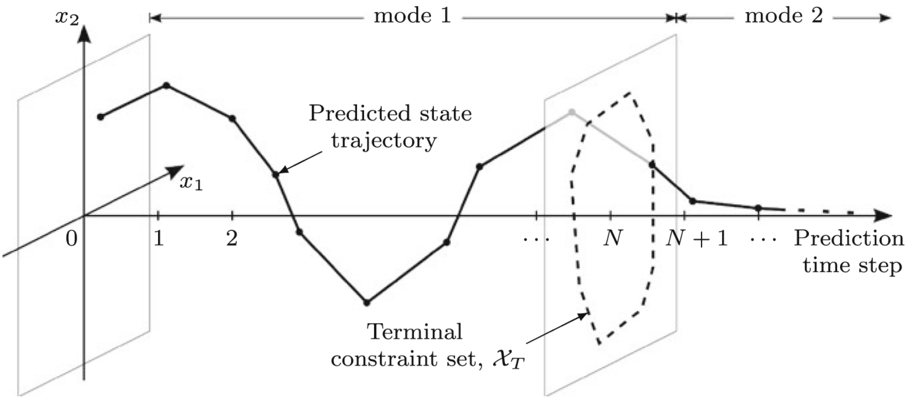
ε = 1e-3

Example



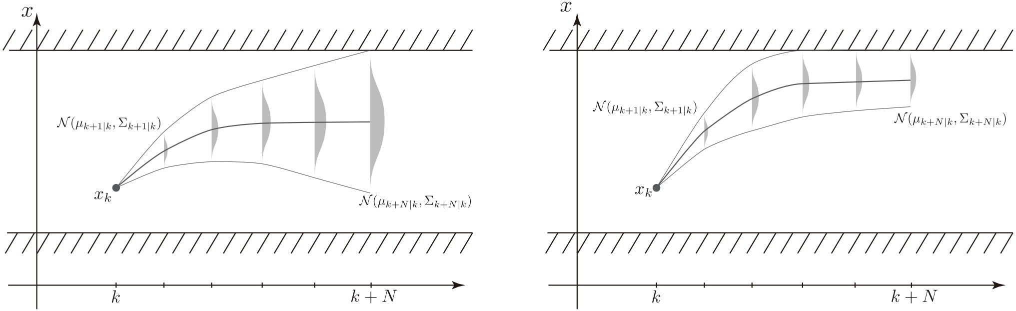


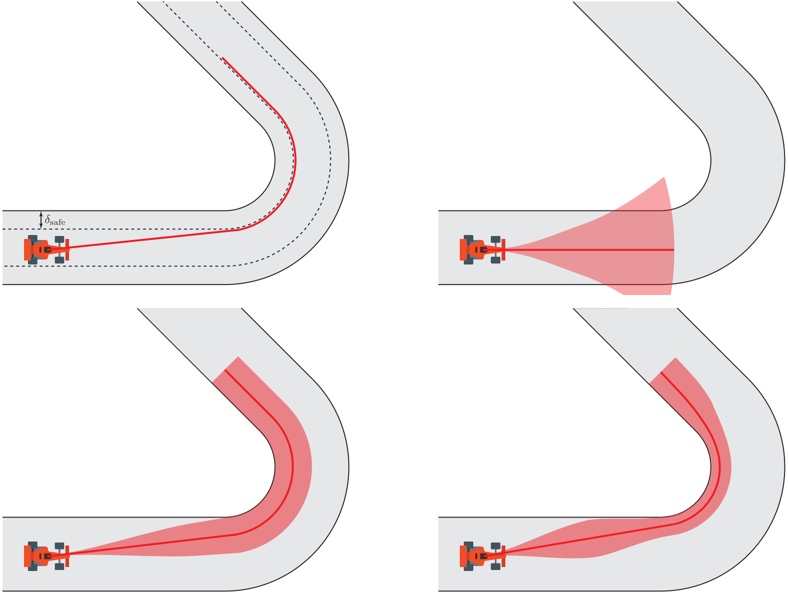
MPC



(*Model Predictive Control: Classical, Robust and Stochastic*, B. Kouvaritakis and M. Cannon)

Stochastic MPC




δ_{safe}

Stochastic MPC

$$\min_{u_{k|k},\ldots,u_{k+N-1|k}} J_N(\mu_k, \Sigma_k; u_{k|k}, \ldots, u_{k+N-1|k}) = \mathbb{E}_k \left[\sum_{t=k}^{k+N-1} x_{t|k}^\top Q x_{t|k} + u_{t|k}^\top R u_{t|k} \right] + J_f(x_{k+N|k})$$

subject to

$$x_{t+1|k} = A x_{t|k} + B u_{t|k} + D w_t, \qquad x_{k|k} = x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$$

$$\Pr_k \left(\alpha_{x,i}^\top x_{t|k} \leq \beta_{x,i} \right) \geq 1 - p_{x,i}, \quad i = 0, \ldots, N_s - 1$$

$$\Pr_k \left(\alpha_{u,j}^\top u_{t|k} \leq \beta_{u,j} \right) \geq 1 - p_{u,j}, \quad j = 0, \ldots, N_c - 1$$

Stochastic MPC

$$\min_{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}} J_N(x_k; u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}) =$$

$$\mathbb{E}_k \left[\sum_{t=k}^{k+N-1} x_{t|k}^\top Q x_{t|k} + u_{t|k}^\top R u_{t|k} \right] + \mathbb{E}_k[x_{k+N|k}]^\top P_{\text{mean}} \mathbb{E}_k[x_{k+N|k}]$$

subject to

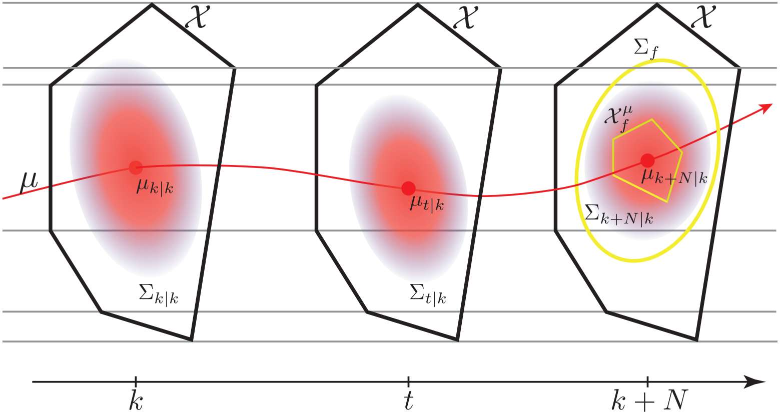
$$x_{t+1|k} = A x_{t|k} + B u_{t|k} + D w_t, \qquad x_{k|k} = x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$$

$$\Pr_k \left(\alpha_{x,i}^\top x_{t|k} \leq \beta_{x,i} \right) \geq 1 - p_{x,i}, \quad i = 0, \ldots, N_s - 1$$

$$\Pr_k \left(\alpha_{u,j}^\top u_{t|k} \leq \beta_{u,j} \right) \geq 1 - p_{u,j}, \quad j = 0, \ldots, N_c - 1$$

$$\mathbb{E}_k \left[x_{k+N|k} \right] \in \mathcal{X}_f^\mu$$

$$\mathbb{E}_k \left[(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])^\top \right] \preceq \Sigma_f$$


$\mathcal{X}$
$\mathcal{X}$
$\mathcal{X}$
Σ_f
$\mathcal{X}_f^\mu$
μ
$\mu_{k|k}$
$\mu_{t|k}$
$\mu_{k+N|k}$
$\Sigma_{k+N|k}$
$\Sigma_{k|k}$
$\Sigma_{t|k}$
k
t
$k+N$

Stochastic MPC

$$\min_{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}} J_N(x_k; u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}) =$$

$$\mathbb{E}_k\left[\sum_{t=k}^{k+N-1} x_{t|k}^\top Q x_{t|k} + u_{t|k}^\top R u_{t|k}\right] + \mathbb{E}_k[x_{k+N|k}]^\top P_{\text{mean}} \mathbb{E}_k[x_{k+N|k}]$$

subject to

$$x_{t+1|k} = Ax_{t|k} + Bu_{t|k} + Dw_t, \qquad x_{k|k} = x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$$

$$\Pr_k\left(\alpha_{x,i}^\top x_{t|k} \leq \beta_{x,i}\right) \geq 1 - p_{x,i}, \quad i = 0, \ldots, N_s - 1$$

$$\Pr_k\left(\alpha_{u,j}^\top u_{t|k} \leq \beta_{u,j}\right) \geq 1 - p_{u,j}, \quad j = 0, \ldots, N_c - 1$$

$$\mathbb{E}_k\left[x_{k+N|k}\right] \in \mathcal{X}_f^\mu$$

$$\mathbb{E}_k\left[(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])^\top\right] \preceq \Sigma_f$$

Theorem

Given μ_k, Σ_k, $\mathcal{X}_f^{\mu}$, $\Sigma_f \succ 0$, *and* $P_{\text{mean}} \succ 0$, *and using the following control law*

$$u_{t|k} = v_{t|k} + K_{t|k} y_{t|k}$$

$$y_{t+1|k} = A y_{t|k} + D w_t$$
$$y_{k|k} = x_{k|k} - \mu_{k|k}$$

the problem can be cast as a convex programming problem

$$
\begin{aligned}
\min_{V,K} J_N(\mu_k, \Sigma_k; V, K) = \mathtt{tr}\left[\left((I+\mathcal{B}K)^\top \bar{Q}_{P,\mathrm{cov}}(I+\mathcal{B}K) + K^\top \bar{R} K\right)\Sigma_y\right] \\
+ (\mathcal{A}\mu_{k|k} + \mathcal{B}V)^\top \bar{Q}_{P,\mathrm{mean}}(\mathcal{A}\mu_{k|k} + \mathcal{B}V) + V^\top \bar{R} V
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\alpha_{x,i}^\top E_{t-k}\left(\mathcal{A}\mu_{k|k} + \mathcal{B}V\right) + \|\Sigma_y^{1/2}(I+\mathcal{B}K)^\top E_{t-k}^\top \alpha_{x,i}\|\Phi^{-1}(1-p_{x,i}) - \beta_{x,i} \le 0 \\
&\alpha_{u,j}^\top F_{t-k} V + \|\Sigma_y^{1/2} K^\top F_{t-k}^\top \alpha_{u,j}\|\Phi^{-1}\left(1-p_{u,j}\right) - \beta_{u,j} \le 0 \\
&E_N\left(\mathcal{A}\mu_{k|k} + \mathcal{B}V\right) \in \mathcal{X}_f^\mu \\
&\Sigma_f \succeq E_N(I+\mathcal{B}K)\Sigma_y(I+\mathcal{B}K)^\top E_N^\top
\end{aligned}
$$

where

$$
\mu_{k|k} = \mu_k, \qquad \Sigma_{k|k} = \Sigma_k, \qquad \Sigma_y = \mathcal{A}\Sigma_{k|k}\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top
$$

$$V = \begin{bmatrix} v_{k|k} \\ \vdots \\ v_{k+N-1|k} \end{bmatrix}, \qquad K = \begin{bmatrix} K_{k|k} & & & & 0 \\ & K_{k+1|k} & & & 0 \\ & & \ddots & & 0 \\ & & & K_{k+N-1|k} & 0 \end{bmatrix}$$

$$\bar{Q}_{P,\text{mean}} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P_{\text{mean}} \end{bmatrix}, \quad \bar{Q}_{P,\text{cov}} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & 0 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}.$$

Note that the terminal covariance constraint

$$\Sigma_f \succeq E_N(I + \mathcal{B}K)\Sigma_y(I + \mathcal{B}K)^\top E_N^\top$$

can be converted to a linear matrix inequality (LMI)

$$\begin{bmatrix} \Sigma_f & E_N(I + \mathcal{B}K)\Sigma_y^{1/2} \\ \Sigma_y^{1/2}(I + \mathcal{B}K)^\top E_N^\top & I \end{bmatrix} \succeq 0$$

Stochastic MPC

$$\min_{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}} J_N(x_k; u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}) =$$

$$\mathbb{E}_k\left[\sum_{t=k}^{k+N-1} x_{t|k}^\top Q x_{t|k} + u_{t|k}^\top R u_{t|k}\right] + \mathbb{E}_k[x_{k+N|k}]^\top P_{\text{mean}} \mathbb{E}_k[x_{k+N|k}]$$

subject to

$$x_{t+1|k} = A x_{t|k} + B u_{t|k} + D w_t, \qquad x_{k|k} = x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$$

$$\Pr_k\left(\alpha_{x,i}^\top x_{t|k} \leq \beta_{x,i}\right) \geq 1 - p_{x,i}, \quad i = 0, \ldots, N_s - 1$$

$$\Pr_k\left(\alpha_{u,j}^\top u_{t|k} \leq \beta_{u,j}\right) \geq 1 - p_{u,j}, \quad j = 0, \ldots, N_c - 1$$

$$\mathbb{E}_k\left[x_{k+N|k}\right] \in \mathcal{X}_f^\mu$$

$$\mathbb{E}_k\left[(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])(x_{k+N|k} - \mathbb{E}[x_{k+N|k}])^\top\right] \preceq \Sigma_f$$

Covariance Assignment

Definition

The state covariance $\Sigma \succ 0$ is *assignable* to the closed-loop system

$$x_{k+1} = (A + B\tilde{K})x_k + Dw_k$$

if Σ satisfies

$$\Sigma = (A + B\tilde{K})\Sigma(A + B\tilde{K})^\top + DD^\top$$

where $\tilde{K}$ is a state-feedback gain.

The set of **assignable state covariances** Σ can be parameterized by the following set of LMIs

$$(I - BB^+)(\Sigma - A\Sigma A^\top - DD^\top)(I - BB^+) = 0$$
$$\Sigma \succ 0, \qquad \Sigma \succeq DD^\top$$

Proposition (Collins and Skelton, 1987)

Let $\Sigma \succ 0$ be an assignable covariance matrix. Then all (stabilizing) assignability state-feedback gains $\tilde{K}$ are parametrized by

$$\tilde{K} = B^{+} \left((\Sigma - DD^{\top})^{1/2} G_1 \begin{bmatrix} I_r & 0 \\ 0 & T \end{bmatrix} G_2^{\top} S^{-1} - A \right) + (I_{n_u} - B^{+}B)Z$$

where T is an arbitrary orthogonal matrix, $SS^{\top} = \Sigma$, Z is an arbitrary matrix, and G_1 and G_2 are defined from the singular-value decompositions

$$\begin{aligned} (I - BB^{+})(\Sigma - DD^{\top})^{1/2} &= L\Lambda G_1^{\top} \\ (I - BB^{+})AS &= L\Lambda G_2^{\top} \end{aligned}$$

where $\Lambda = \texttt{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.

Theorem

Suppose that Σ_f is assignable, $\mu_f \in \mathcal{X}_f^{\mu}$, where $\mathcal{X}_f^{\mu}$ is a positively invariant, such that for all $\mu \in \mathcal{X}_f^{\mu}$

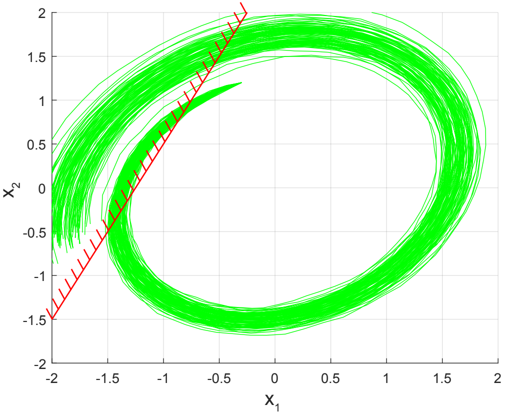
$$
\begin{aligned}
&(A + B\tilde{K})\mu \in \mathcal{X}_f^{\mu} \\
&\alpha_{x,i}^{\top}\mu + \|\Sigma_f{}^{1/2}\alpha_{x,i}\|\Phi^{-1}(1 - p_{x,i}) - \beta_{x,i} \leq 0, \quad i = 0, \ldots, N_s - 1 \\
&\alpha_{u,j}^{\top}\tilde{K}\mu + \|\Sigma_f{}^{1/2}\tilde{K}^{\top}\alpha_{u,j}\|\Phi^{-1}(1 - p_{u,j}) - \beta_{u,j} \leq 0, \quad j = 0, \ldots, N_c - 1
\end{aligned}
$$

where $\tilde{K}$ is from corresponding assignability gain matrix, and P_{mean} is the solution of the discrete-time Lyapunov equation

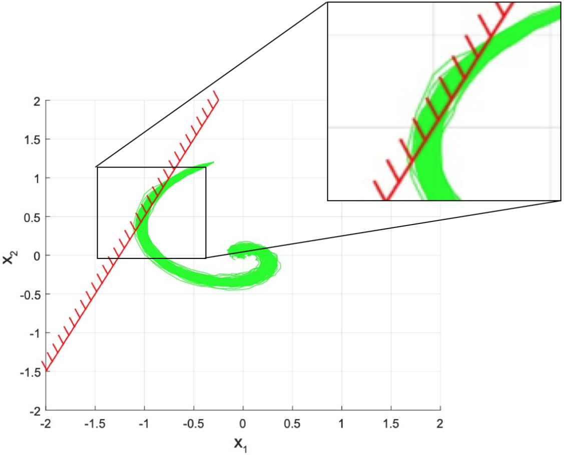
$$(A + B\tilde{K})^{\top} P_{\text{mean}} (A + B\tilde{K}) - P_{\text{mean}} + Q + \tilde{K}^{\top} R \tilde{K} = 0$$

Then, the solution ensures recursive feasibility and stability.

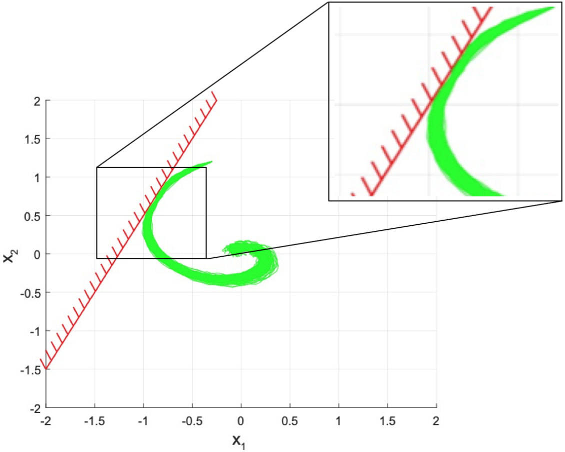
Uncontrolled



Infinite Horizon LQR



CS-SMPC



For More Details...

- Goldshtein, M., and Tsiotras, P., "Finite-Horizon Covariance Control of Linear Time-Varying Systems," *56th IEEE Conference on Decision and Control*, Melbourne, Australia, Dec. 12–15, 2017, pp. 3606–3611.
- Ridderhof, J., and Tsiotras, P., "Uncertainty Quantification and Control During Mars Powered Descent and Landing using Covariance Steering," *AIAA Guidance, Navigation, and Control Conference*, (AIAA 2018-1576), Kissimmee, FL, Jan. 8–12, 2018.
- Okamoto, K., Goldshtein, M., and Tsiotras, P., "Optimal Covariance Control for Stochastic Systems Under Chance Constraints," *IEEE Control Systems Letters*, Vol. 2, No. 2, pp. 266–271, 2018.
- Okamoto, K. and Tsiotras, P., "Optimal Stochastic Vehicle Path Planning Using Covariance Steering," *IEEE Robotics and Automation Letters*, Vol. 4, No. 3, pp. 2276–2281, 2019,
- Riderhoff, J., and Tsiotras, P., "Minimum-fuel Powered Descent in the Presence of Random Disturbances," *AIAA Guidance, Navigation, and Control Conference*, San Diego, CA, Jan. 7–11, 2019 (**best student paper award**).

For More Details...

- Okamoto, K., and Tsiotras, P., "Stochastic Model Predictive Control for Constrained Linear Systems Using Optimal Covariance Steering," http://arxiv.org/abs/1905.13296
- Okamoto, K. and Tsiotras, P., "Input Hard Constrained Optimal Covariance Steering," *58th IEEE Conference on Decision and Control*, Nice, France, Dec. 11–13, 2019. Session ThA19.3 Stochastic Systems II
- Ridderhof, J., Okamoto, K. and Tsiotras, P., "Nonlinear Uncertainty Control with Iterative Covariance Steering," *58th IEEE Conference on Decision and Control*, Nice, France, Dec. 11–13, 2019. Session ThA19.3 Stochastic Systems II

Many Extensions

- Output feedback covariance steering (Bakolas, 2019; Ridderhof and PT, 2020; Maity and PT, 2020)
- Input constrains (Okamoto and PT, 2019); see paper in Session ThA19.3 Stochastic Systems II
- Extension to nonlinear systems (Caluya and Halder 2019; Ridderhof, Okamoto, and PT, 2019); see paper in Session ThA19.3 Stochastic Systems II
- Differential games (Makkapatti, Okamoto and PT, 2019)